

# THREE-DIMENSIONAL CENTRAL CONFIGURATIONS IN $\mathbb{H}^3$ AND $\mathbb{S}^3$

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**ABSTRACT.** We show that each central configuration in the three-dimensional hyperbolic sphere is equivalent to one central configuration on a particular two-dimensional hyperbolic sphere. However, there exist both special and ordinary central configurations in the three-dimensional sphere that are not confined to any two-dimensional sphere.

**Key Words:** celestial mechanics; curved  $N$ -body problem; central configurations.

## 1. INTRODUCTION

The curved  $N$ -body problem is a natural extension of the Newtonian  $N$ -body problem in  $\mathbb{R}^3$  to isotropic, complete, simply connected spaces of constant nonzero curvature,  $\mathbb{S}^3$  and  $\mathbb{H}^3$ . For its history, we refer the readers to [3], where the equations of motion are written in extrinsic coordinates in  $\mathbb{R}^4$  for  $\mathbb{S}^3$ , and the Minkowski space  $\mathbb{R}^{3,1}$  for  $\mathbb{H}^3$ . This approach, different from more traditional ones like [9], led to fruitful results, especially in the study of relative equilibria, which are rigid motions that become fixed points in some rotating coordinates system, [2, 3, 4, 5, 6, 7].

Based on the work of Diacu, especially [3, 5], the authors of [8] proposed to study central configurations. Roughly speaking, central configurations are special arrangements of the point particles and the exact definition will be given later. The central configurations of the Newtonian  $N$ -body problem, first formulated by Laplace [10], are quite important in the study of the Newtonian  $N$ -body problem. In [8], the authors have also showed the importance of central configurations for the curved  $N$ -body problem. For instance, each central configuration gives rise to

a one-parameter family of relative equilibria, and central configurations are the bifurcation points in the topological classification of the curved  $N$ -body problem.

Some questions about these configurations were also raised in [8]. For example, find all central configurations for  $N$  point particles when  $N$  is small (the three-particles case has been recently solved and will appear in a forthcoming paper). Another interesting problem is to prove (or disprove) that for generic  $N$  point particles, the number of equivalent classes of central configurations is finite. Though these questions are similar to those of the Newtonian  $N$ -body problem [11, 12], the answers are quite different in general. For example, Moulton's theorem concerning the collinear central configurations has been generalized to  $\mathbb{H}^3$ , [8], but it can not be directly generalized to  $S^3$ .

In this paper, we put into the evidence another difference: each central configuration on  $\mathbb{H}^3$  is equivalent to one central configuration on  $\mathbb{H}_{xyw}^2$ , which will be defined later, whereas in  $S^3$  there are central configurations that are not confined to any two-dimensional sphere. In some sense, the number of central configurations in  $\mathbb{H}^3$  is smaller than that of  $S^3$ . When we consider the Wintner-Smale conjecture in  $\mathbb{H}^3$  raised in [8] asking whether the number of classes of central configurations in  $\mathbb{H}^3$  is finite or not for generic  $N$  point particles, we only need to study the problem on  $\mathbb{H}_{xyw}^2$ .

The paper is organized as follows: in Section 2, we recall the basic setting of the curved  $N$ -body problem and the corresponding facts about central configurations; in Section 3, we prove the result about central configurations in  $\mathbb{H}^3$ ; in Section 4, we construct a two-parameter family of three-dimensional central configurations in  $S^3$ .

## 2. THE CURVED $N$ -BODY PROBLEM AND CENTRAL CONFIGURATIONS

**2.1. Equations of motion.** As done in [3, 5], the equations will be written in  $\mathbb{R}^4$  for  $S^3$  and in the Minkowski space  $\mathbb{R}^{3,1}$ , for  $\mathbb{H}^3$ . For convenience, we understand the two linear spaces as  $\mathbb{R}^4$  endowed with two inner products: for two vectors,  $\mathbf{q}_1 = (x_1, y_1, z_1, w_1)^T$  and  $\mathbf{q}_2 = (x_2, y_2, z_2, w_2)^T$ , the inner products are given by

$$\mathbf{q}_1 \cdot \mathbf{q}_2 = x_1x_2 + y_1y_2 + z_1z_2 + \sigma w_1w_2,$$

where  $\sigma = 1$  for the Euclidean space and  $\sigma = -1$  for the Minkowski space. We define the unit sphere  $S^3$  and the unit hyperbolic sphere  $\mathbb{H}^3$  as

$$\begin{aligned} S^3 &:= \{(x, y, z, w)^T \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\} \quad \text{and} \\ \mathbb{H}^3 &:= \{(x, y, z, w)^T \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 - w^2 = -1, w > 0\}, \end{aligned}$$

respectively. We can merge these two manifolds into

$$M^3 := \{(x, y, z, w)^T \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + \sigma w^2 = \sigma, \text{ with } w > 0 \text{ for } \sigma = -1\}.$$

Given the positive masses  $m_1, \dots, m_N$ , whose positions are described by the configuration  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N) \in (\mathbb{M}^3)^N$ ,  $\mathbf{q}_i = (x_i, y_i, z_i, w_i)^T$ ,  $i = \overline{1, N}$ , we define the singularity set

$$\Delta = \cup_{1 \leq i < j \leq N} \{\mathbf{q} \in (\mathbb{M}^3)^N; \mathbf{q}_i \cdot \mathbf{q}_j = \pm 1\}.$$

Let  $d_{ij}$  be the geodesic distance between the point masses  $m_i$  and  $m_j$ , which is computed by

$$d_{ij}(\mathbf{q}) = \arccos(\mathbf{q}_i \cdot \mathbf{q}_j) \text{ for } \mathbb{S}^3, \quad d_{ij}(\mathbf{q}) = \operatorname{arccosh}(-\mathbf{q}_i \cdot \mathbf{q}_j) \text{ for } \mathbb{H}^3.$$

The force function  $U$  ( $-U$  being the potential function) in  $(\mathbb{M}^3)^N \setminus \Delta$  is

$$U(\mathbf{q}) := \sum_{1 \leq i < j \leq N} m_i m_j \operatorname{ctnd}_{ij}(\mathbf{q}),$$

where  $\operatorname{ctn}(x)$  stands for  $\cot(x)$  in  $\mathbb{S}^3$  and  $\operatorname{coth}(x)$  in  $\mathbb{H}^3$ . We also introduce two more notations, which unify the trigonometric and hyperbolic functions,

$$\operatorname{sn}(x) = \sin(x) \text{ or } \sinh(x), \quad \operatorname{csn}(x) = \cos(x) \text{ or } \cosh(x).$$

Define the kinetic energy as  $T(\dot{\mathbf{q}}) = \sum_{1 \leq i \leq N} m_i \dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i$ ,  $\dot{\mathbf{q}} = (\dot{\mathbf{q}}_1, \dots, \dot{\mathbf{q}}_N)$ . Then the curved  $N$ -body problem is given by the Lagrange system on  $T((\mathbb{M}^3)^N \setminus \Delta)$ , with

$$L(\mathbf{q}, \dot{\mathbf{q}}) := T(\dot{\mathbf{q}}) + U(\mathbf{q}).$$

Using variational methods, we obtain the equations of motion in  $\mathbb{S}^3$  and in  $\mathbb{H}^3$ , [8]. Merged into one, they are

$$\begin{cases} \ddot{\mathbf{q}}_i = \sum_{j=1, j \neq i}^N \frac{m_i m_j [\mathbf{q}_j - \operatorname{csnd}_{ij} \mathbf{q}_i]}{\operatorname{sn}^3 d_{ij}} - \sigma m_i (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) \mathbf{q}_i \\ \mathbf{q}_i \cdot \mathbf{q}_i = \sigma, \quad i = \overline{1, N}. \end{cases}$$

The first part of the acceleration access from the gradient of the force function,  $\nabla_{\mathbf{q}_i} U(\mathbf{q})$ , and we will denote it by  $\mathbf{F}_i$ . It is the sum of  $\mathbf{F}_{ij} = \frac{m_i m_j [\mathbf{q}_j - \operatorname{csnd}_{ij} \mathbf{q}_i]}{\operatorname{sn}^3 d_{ij}}$  for  $j \neq i$ .

## 2.2. Central configurations.

**Definition 1.** A configuration  $\mathbf{q} \in (\mathbb{M}^3)^N \setminus \Delta$  is called a central configuration if there is some constant  $\lambda$  such that

$$(1) \quad \nabla_{\mathbf{q}_i} U(\mathbf{q}) = \lambda \nabla_{\mathbf{q}_i} I(\mathbf{q}), \quad i = \overline{1, N},$$

where  $\nabla$  is the gradient operator in  $\mathbb{M}^3$ ,  $I(\mathbf{q}) = \sum_{i=1}^N m_i (x_i^2 + y_i^2)$ , and the explicit form of  $\nabla_{\mathbf{q}_i} I$  is

$$(2) \quad 2m_i \begin{bmatrix} x_i(w_i^2 + z_i^2) \\ y_i(w_i^2 + z_i^2) \\ -z_i(x_i^2 + y_i^2) \\ -w_i(x_i^2 + y_i^2) \end{bmatrix} \text{ in } T(S^3)^N \quad \text{and} \quad 2m_i \begin{bmatrix} x_i(w_i^2 - z_i^2) \\ y_i(w_i^2 - z_i^2) \\ z_i(x_i^2 + y_i^2) \\ w_i(x_i^2 + y_i^2) \end{bmatrix} \text{ in } T(H^3)^N.$$

Since the two functions  $U$  and  $I$  are both invariant under the group action of  $SO(2) \times SO(2)$  (in the case of  $S^3$ ) and  $SO(2) \times SO(1,1)$  (in the case of  $H^3$ ), it is easy to check that a central configuration remains a central configuration after an  $SO(2) \times SO(2)$  action (or an  $SO(2) \times SO(1,1)$  action), [8]. Two central configurations are said to be *equivalent* if one can be transformed to the other by these group actions. When we say a central configuration, we mean a class of central configurations as defined by the equivalence relation.

A central configuration with  $\lambda = 0$  is called a *special central configuration*, which only occurs in  $S^3$ , [3]. Otherwise, it is called an *ordinary central configuration*. A central configuration lying on a geodesic is called a *geodesic central configuration*. A central configuration lying on a two-dimensional sphere is called an  $S^2$  *central configuration*, a central configuration lying on a two-dimensional hyperbolic sphere is called an  $H^2$  *central configuration*. All the other central configurations are called *three-dimensional central configurations*.

Here, a two-dimensional sphere (hyperbolic sphere) means a sphere (hyperbolic sphere) isometric to the unit sphere (hyperbolic sphere) in  $\mathbb{R}^3$  ( $\mathbb{R}^{2,1}$ ). It is the non-empty intersection of  $\mathbb{M}^3$  with a three-dimensional linear subspace:  $\{(x, y, z, w)^T \in \mathbb{R}^4 | ax + by + cz + dw = 0\}$ , [1]. We begin with the following result.

**Proposition 1.** *Let  $V = \{(x, y, z, w)^T \in \mathbb{R}^4 | cz + \sigma dw = 0\}$ . If a configuration  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$  lies on the two-dimensional sphere (hyperbolic sphere):  $V \cap \mathbb{M}^3$ , then  $\nabla_{\mathbf{q}_i} I$  is in  $V$  for  $i = \overline{1, N}$ .*

*Proof.* By the explicit form of  $\nabla_{\mathbf{q}_i} I$ , equation (2), we get

$$\nabla_{\mathbf{q}_i} I \cdot (0, 0, c, d)^T = -\sigma 2m_i(x_i^2 + y_i^2)(cz_i + \sigma dw_i) = 0.$$

This equation completes the proof.  $\square$

### 3. CENTRAL CONFIGURATIONS ON $H^3$

Let us define  $H_{xyw}^2 := \{(x, y, z, w)^T \in \mathbb{R}^4 | z = 0\} \cap H^3$ . We can prove the following result.

**Theorem 1.** *Each central configuration in  $H^3$  is equivalent to some central configuration on  $H_{xyw}^2$ .*

*Proof.* We first show that all central configurations in  $H^3$  must lie on a two-dimensional hyperbolic sphere. Then we show that there is some action  $\chi \in$

$SO(2) \times SO(1, 1)$  which transforms that hyperbolic sphere to  $H_{xyw}^2$ . Thus by the definition of equivalent central configurations, each central configuration in  $H^3$  is equivalent to some central configuration on  $H_{xyw}^2$ .

Consider the two-dimensional hyperbolic sphere:  $H_\phi^2 := \{(x, y, z, w)^T \in \mathbb{R}^4 \mid \cosh \phi z - \sinh \phi w = 0\} \cap H^3$ . The intersection is not empty, since the linear subspace and  $H^3$  share the point  $(0, 0, \sinh \phi, \cosh \phi)^T$ . We show that each central configuration will be confined to only one such two-dimensional hyperbolic sphere.

Assume that this is not the case. Suppose that there is a central configuration  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$  with  $\mathbf{q}_i \in H_{\phi_i}^2$ ,  $\phi_1 \geq \phi_i$  for  $i \neq 1$  and there is at least one  $i$  such that  $\phi_1 > \phi_i$ . Then  $\mathbf{q}_i$  can be written as  $(x_i, y_i, \rho_i \sinh \phi_i, \rho_i \cosh \phi_i)^T$  with  $\rho_i > 0$  since  $w_i = \rho_i \cosh \phi_i > 0$ . By Proposition 1,  $\nabla_{\mathbf{q}_1} I$  is in the linear subspace  $\{(x, y, z, w)^T \in \mathbb{R}^4 \mid \cosh \phi_1 z - \sinh \phi_1 w = 0\}$ . In order to have a central configuration,  $\nabla_{\mathbf{q}_1} U$  must be in the linear subspace, i.e.,

$$\nabla_{\mathbf{q}_1} U \cdot (0, 0, \cosh \phi_1, \sinh \phi_1)^T = \mathbf{F}_{1z} \cosh \phi_1 - \mathbf{F}_{1w} \sinh \phi_1 = 0,$$

where  $\mathbf{F}_{1z}$  and  $\mathbf{F}_{1w}$  stand for the  $z$ -coordinate and  $w$ -coordinate of  $\mathbf{F}_1$ , respectively. However, using the explicit form of  $\mathbf{F}_1$ , we get

$$\begin{aligned} & \mathbf{F}_{1z} \cosh \phi_1 - \mathbf{F}_{1w} \sinh \phi_1 \\ &= \sum_{i=2}^N m_i m_1 \left( \frac{z_i - \cosh d_{i1} z_1}{\sinh^3 d_{i1}} \cosh \phi_1 - \frac{w_i - \cosh d_{i1} w_1}{\sinh^3 d_{i1}} \sinh \phi_1 \right) \\ &= \sum_{i=2}^N m_i m_1 \frac{\rho_i \sinh \phi_i \cosh \phi_1 - \rho_i \cosh \phi_i \sinh \phi_1 - \cosh d_{i1} (z_1 \cosh \phi_1 - w_1 \sinh \phi_1)}{\sinh^3 d_{i1}} \\ &= \sum_{i=2}^N m_i m_1 \frac{\rho_i \sinh(\phi_i - \phi_1)}{\sinh^3 d_{i1}} < 0, \end{aligned}$$

since  $\phi_i \leq \phi_1$  for  $i \neq 1$  and there is at least one  $i$  such that  $\phi_i < \phi_1$ .

Thus any central configuration must lie on only one such hyperbolic sphere, say  $H_\phi^2$ . Let

$$\chi = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \cosh \phi & -\sinh \phi \\ -\sinh \phi & \cosh \phi \end{bmatrix} \right) \in SO(2) \times SO(1, 1).$$

Since  $\begin{bmatrix} \cosh \phi & -\sinh \phi \\ -\sinh \phi & \cosh \phi \end{bmatrix} \begin{bmatrix} \rho_i \sinh \phi \\ \rho_i \cosh \phi \end{bmatrix} = \begin{bmatrix} 0 \\ \rho_i \end{bmatrix}$ ,  $\chi(H_\phi^2) = H_{xyw}^2$ . This calculation completes the proof.  $\square$

To offer more insight into this result, we provide a heuristic argument. Recall that the Poincaré ball model of  $\mathbb{H}^3$  is

$$\left( \bar{x}^2 + \bar{y}^2 + \bar{z}^2 < 1, \quad ds^2 = \frac{4(d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2)}{(1 - \bar{x}^2 - \bar{y}^2 - \bar{z}^2)^2} \right).$$

In this model, a two-dimensional hyperbolic sphere is the intersection of the three-dimensional ball with a two-dimensional Euclidean sphere that orthogonally intersects the boundary of the ball. The hyperbolic spheres  $\mathbb{H}_\phi^2$  defined above are those that intersect the  $\bar{z}$ -axis orthogonally, [1]. For example,  $\mathbb{H}_{xyw}^2$  in this model is the disk in the plane  $\bar{z} = 0$ . Now suppose that  $\mathbf{q}_i \in \mathbb{H}_{\phi_i}^2$  and  $\phi_1 > \phi_2$ . Proposition 1 implies that  $\nabla_{\mathbf{q}_1} I \in T_{\mathbf{q}_1} \mathbb{H}_{\phi_1}^2$ , as showed in Figure 1. However,  $\mathbf{F}_{12}$  points towards the lower hyperbolic sphere  $\mathbb{H}_{\phi_2}^2$ . Thus  $\nabla_{\mathbf{q}_1} I$  and  $\nabla_{\mathbf{q}_1} U$  cannot be collinear.

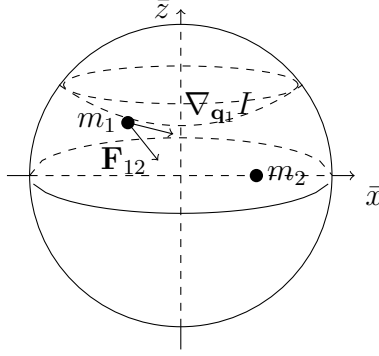


FIGURE 1. A configurations in the Poincaré ball model.

**Remark 1.** Recall that there are central configurations not in a plane (called spatial central configurations) in the Newtonian  $N$ -body problem, such as the regular tetrahedron for any given four masses. However, those spatial configurations do not lead to rigid motions. Thus if we defined central configurations in  $\mathbb{R}^3$  as those that lead to rigid motions, there would be no spatial ones.

#### 4. CENTRAL CONFIGURATIONS IN $S^3$

Apparently, the compactness of  $S^3$  makes the set of central configuration in it richer than in  $\mathbb{H}^3$ . With computations similar to the ones we performed in  $\mathbb{H}^3$ , we can get the following necessary conditions for central configurations in  $S^3$ ,

$$\sum_{j=1, j \neq i}^N m_i m_j \frac{\rho_j \sin(\phi_j - \phi_i)}{\sin^3 d_{ij}} = 0, \quad i = \overline{1, N}.$$

These equations, however, do not rule out the existence of three-dimensional central configurations. For example, we have the pentatope special central configuration of five equal masses, [8],

$$\begin{aligned}
x_1 &= 1, & y_1 &= 0, & z_1 &= 0, & w_1 &= 0, \\
x_2 &= -1/4, & y_2 &= \sqrt{15}/4, & z_2 &= 0, & w_2 &= 0, \\
x_3 &= -1/4, & y_3 &= -\sqrt{5}/(4\sqrt{3}), & z_3 &= \sqrt{5}/\sqrt{6}, & w_3 &= 0, \\
x_4 &= -1/4, & y_4 &= -\sqrt{5}/(4\sqrt{3}), & z_4 &= -\sqrt{5}/(2\sqrt{6}), & w_4 &= \sqrt{5}/(2\sqrt{2}), \\
x_5 &= -1/4, & y_5 &= -\sqrt{5}/(4\sqrt{3}), & z_5 &= -\sqrt{5}/(2\sqrt{6}), & w_5 &= -\sqrt{5}/(2\sqrt{2}).
\end{aligned}$$

However, all known three-dimensional central configurations are special central configurations (i.e.,  $\lambda = 0$ ). We will further construct a two-parameter family of ordinary three-dimensional central configurations of five masses. Suppose that the masses are  $m_1 = m_2 = m, m_3 = m_4 = m_5 = 1$ , and their positions are given by

$$\begin{aligned}
x_1 &= 0, & y_1 &= 0, & z_1 &= \cos \theta, & w_1 &= \sin \theta, \\
x_2 &= 0, & y_2 &= 0, & z_2 &= \cos \theta, & w_2 &= -\sin \theta, \\
x_3 &= r, & y_3 &= 0, & z_3 &= c, & w_3 &= 0, \\
x_4 &= r \cos \frac{2\pi}{3}, & y_4 &= r \sin \frac{2\pi}{3}, & z_4 &= c, & w_4 &= 0, \\
x_5 &= r \cos \frac{4\pi}{3}, & y_5 &= r \sin \frac{4\pi}{3}, & z_5 &= c, & w_5 &= 0,
\end{aligned}$$

where  $c \in (-1, 1) \setminus \{0\}$ ,  $r > 0$ ,  $r^2 + c^2 = 1$  and  $\theta \in (0, \pi) \setminus \{\frac{\pi}{2}\}$ . Such configurations depend on two parameters,  $c$  and  $\theta$ , and we denote them by  $\mathbf{q}(c, \theta)$ . It is easy to see that these configurations are not confined to any two-dimensional sphere. In Figure 2, we illustrate such a configuration in a  $\mathbb{R}^3$  hyperplane by the stereographic projection of  $S^3$  from  $(0, 0, 1, 0)$  onto the corresponding equatorial  $\mathbb{R}^3$  hyperplane, i.e.,

$$\bar{x} = \frac{x}{1-z}, \quad \bar{y} = \frac{y}{1-z}, \quad \bar{w} = \frac{w}{1-z}.$$

**Proposition 2.** *For any  $(c, \theta) \in (-1, 0) \times (0, \frac{\pi}{2})$ , and  $(c, \theta) \in (0, 1) \times (\frac{\pi}{2}, \pi)$ , the configurations  $\mathbf{q}(c, \theta)$  constructed above are central configurations if*

$$(3) \quad m = -\frac{3c|\sin^3 2\theta|}{2\cos\theta(1-c^2\cos^2\theta)^{3/2}}.$$

*Generally, they are ordinary central configurations.*

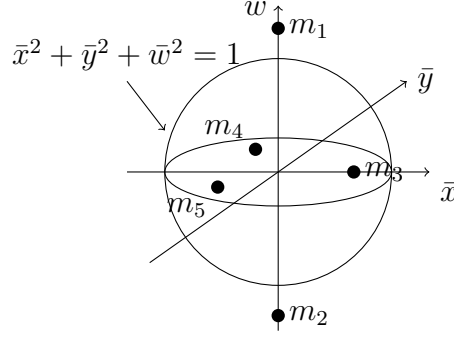


FIGURE 2. A configuration  $\mathbf{q}(c, \theta)$  with  $(c, \theta) \in (-1, 0) \times (0, \frac{\pi}{2})$ .

*Proof.* We check that the central configuration equations  $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I, i = 1, \dots, 5$ , are satisfied. The function  $U$  can be written as  $U = U_1 + U_2$ , where

$$U_1 = \cot d_{34} + \cot d_{45} + \cot d_{35}, \quad U_2 = m^2 \cot d_{12} + m \sum_{i=3}^5 (\cot d_{1i} + \cot d_{2i}).$$

Note that the three equal masses  $m_3, m_4$ , and  $m_5$  form an ordinary central configuration themselves, i.e.,  $\nabla_{\mathbf{q}_i} U_1 = \lambda_1 \nabla_{\mathbf{q}_i} I$ , for  $i = 3, 4, 5$ ,  $\lambda_1 = \frac{-3}{2 \sin^3 d_{34}}$ , [8]. Note that  $\nabla_{\mathbf{q}_1} I = \nabla_{\mathbf{q}_2} I = 0$  by equation (2). Thus  $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I$  is satisfied if and only if there is some constant  $\lambda_2$  such that

$$\nabla_{\mathbf{q}_i} U_2 = \lambda_2 \nabla_{\mathbf{q}_i} I, i = 3, 4, 5, \quad \text{and} \quad \mathbf{F}_1 = \mathbf{F}_2 = 0.$$

By symmetry, we only need to check  $\nabla_{\mathbf{q}_3} U_2 = \lambda_2 \nabla_{\mathbf{q}_3} I$ , and  $\mathbf{F}_1 = 0$ .

Note that  $d_{13} = d_{23} = d_{14} = d_{24} = d_{15} = d_{25}$ ,  $d_{34} = d_{45} = d_{35}$ , and

$$\cos d_{12} = \cos 2\theta, \quad \cos d_{13} = c \cos \theta, \quad \cos d_{34} = \frac{3}{2}c^2 - \frac{1}{2}.$$

Some straightforward computation shows

$$\begin{aligned} \nabla_{\mathbf{q}_3} U_2 &= \mathbf{F}_{31} + \mathbf{F}_{32} = \frac{m(\mathbf{q}_1 - \cos d_{13} \mathbf{q}_3)}{\sin^3 d_{13}} + \frac{m(\mathbf{q}_2 - \cos d_{23} \mathbf{q}_3)}{\sin^3 d_{23}} \\ &= \frac{m}{\sin^3 d_{13}} (\mathbf{q}_1 + \mathbf{q}_2 - 2 \cos d_{13} \mathbf{q}_3) = \frac{m}{\sin^3 d_{13}} ((0, 0, 2 \cos \theta, 0)^T - 2c \cos \theta (r, 0, c, 0)^T) \\ &= \frac{-2mr \cos \theta}{\sin^3 d_{13}} (c, 0, -r, 0)^T \end{aligned}$$

Using equation (2), we obtain  $\nabla_{\mathbf{q}_3} I = 2rc(c, 0, -r, 0)^T$ . Thus we can write that

$$\nabla_{\mathbf{q}_3} U_2 = \lambda_2 \nabla_{\mathbf{q}_3} I, \quad \lambda_2 = \frac{-m \cos \theta}{c \sin^3 d_{13}}.$$



By direct computation, we obtain

$$\begin{aligned}
\mathbf{F}_1 &= \mathbf{F}_{12} + \sum_{j=3}^5 \mathbf{F}_{1j} = \frac{m^2}{|\sin^3 2\theta|} (\mathbf{q}_2 - \cos 2\theta \mathbf{q}_1) + \sum_{i=3}^5 \frac{m}{\sin^3 d_{13}} (\mathbf{q}_i - \cos d_{13} \mathbf{q}_1) \\
&= \frac{m^2}{|\sin^3 2\theta|} (\mathbf{q}_2 - \cos 2\theta \mathbf{q}_1) + \frac{m}{\sin^3 d_{13}} \left( \sum_{i=3}^5 \mathbf{q}_i - 3c \cos \theta \mathbf{q}_1 \right) \\
&= m \sin \theta \left( \frac{2m \cos \theta}{|\sin^3 2\theta|} + \frac{3c}{\sin^3 d_{13}} \right) (0, 0, \sin \theta, -\cos \theta)^T.
\end{aligned}$$

Thus  $\mathbf{F}_1 = 0$  if and only if  $m = -\frac{3c|\sin^3 2\theta|}{2 \cos \theta (1-c^2 \cos^2 \theta)^{3/2}}$ . Since we need positive masses,  $c \cos \theta$  needs to be negative.

We have thus obtained a two-parameter family of central configurations  $\mathbf{q}(c, \theta)$  for any  $(c, \theta) \in (-1, 0) \times (0, \frac{\pi}{2})$ , and  $(c, \theta) \in (0, 1) \times (\frac{\pi}{2}, \pi)$ . The central configuration equations  $\nabla_{\mathbf{q}_i} U = \lambda(c, \theta) \nabla_{\mathbf{q}_i} I, i = 1, \dots, 5$ , are satisfied, and the constant is

$$\begin{aligned}
\lambda(c, \theta) &= \lambda_1 + \lambda_2 = \frac{-3}{2 \sin^3 d_{34}} - \frac{m \cos \theta}{c \sin^3 d_{13}} = \frac{-3}{2 \sin^3 d_{34}} + \frac{3|\sin^3 2\theta|}{2 \sin^6 d_{13}} \\
&= \frac{3}{2} \left( \frac{-8}{3\sqrt{3}(1+3c^2)^{3/2}(1-c^2)^{3/2}} + \frac{|\sin^3 2\theta|}{(1-c^2 \cos^2 \theta)^3} \right),
\end{aligned}$$

which is zero on a one-dimensional manifold. Factually, it is homeomorphic to two open unit intervals. Thus generally,  $\mathbf{q}(c, \theta)$  are ordinary central configurations. This remark completes the proof.  $\square$

Moreover, if  $(c, \theta) \in (-1, 0) \times (0, \frac{\pi}{2})$ , then the masses  $m_3, m_4, m_5$  are contained in the unit ball,  $\bar{x}^2 + \bar{y}^2 + \bar{w}^2 \leq 1$ , and the masses  $m_1, m_2$  are outside, see Figure 2. This happens because

$$\bar{w}_1 = \frac{w_1}{1-z_1} = \frac{\sin \theta}{1-\cos \theta} > 1, \quad \bar{x}_3^2 + \bar{y}_3^2 = \left( \frac{x_3}{1-z_3} \right)^2 + \left( \frac{y_3}{1-z_3} \right)^2 = \frac{1+c}{(1-c)} < 1.$$

Similarly, if  $(c, \theta) \in (0, 1) \times (\frac{\pi}{2}, \pi)$ , then masses  $m_3, m_4, m_5$  are outside, but the masses  $m_1, m_2$  are inside the ball.

Obviously, we can still obtain central configurations if we substitute the equilateral triangle by a regular  $n$ -gon with equal masses, and generally they are ordinary ones.

**Acknowledgements.** The authors thank F. Diacu for reading the original manuscript and making many useful suggestions. Suo Zhao is supported by NSFC 11501530 and NSFC 11571242. Shuqiang Zhu is funded by a University of Victoria Scholarship and a David and Geoffery Fox Graduate Fellowship.

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